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ALLOWANCE FOR COMPLEX LOADING IN TRANSVERSELY ISOTROPIC SOLIDS

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A model of plasticity for a transversely isotropic material with allowance for complex loading is developed, based on results of experiments with homogeneous cylindrical specimens of isotropic materials. An empirical model of plasticity for isotropic metals is constructed with allowance for vector properties of the material. The model is extended to a particular case of anisotropy.

Key words: *plasticity, loading process, vector, trace of delay, anisotropy.*

1. Model of Plasticity for Isotropic Metals. According to the isotropy postulate [1], the process of loading and deformation at an arbitrary point of the body is determined by defining five components of the stress and strain vectors, and the process intensity and direction depend only on the internal geometry of the strain path. The material is assumed to be plastically incompressible. The stress vector is presented in the orthonormalized local Frenet reference frame in the following form:

$$\boldsymbol{\sigma} = P_n \boldsymbol{q}_n$$

($\boldsymbol{q}_n$ are the unit Frenet vectors).

In the general case, the components of the Frenet reference frame are determined by the formula

$$\frac{d\boldsymbol{q}_i}{ds} = -\chi_{i-1}\boldsymbol{q}_{i-1} + \chi_i \boldsymbol{q}_{i+1}.$$

Here $\boldsymbol{q}_1 = d\ni/ds$; $\chi_0 = \chi_5 = 0$; χ_1, χ_2, χ_3, and χ_4 are the parameters of curvature and twisting of the strain path.

In terms of velocities, the above-given presentation acquires the form [2]

$$\dot{\boldsymbol{\sigma}} = A_1\boldsymbol{q}_1 + A_2\hat{\boldsymbol{\sigma}} + A_i\boldsymbol{q}_i \qquad (i = 3, 4, 5),$$

where A_i are plasticity functionals and $\hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma}/|\boldsymbol{\sigma}|$.

Constructing the functionals is a complicated task; therefore, hypotheses within the framework of Il'yushin's theory of the processes are used. One of them is the hypothesis about the complanarity of the vectors $\hat{\boldsymbol{\sigma}}$, $\dot{\boldsymbol{\sigma}}$, and $\boldsymbol{q}_1$. In this case, we have $A_3 = A_4 = A_5 = 0$. Some theories of plasticity based on the associated law of the flow can also be constructed within the framework of the complanarity hypothesis. Let the vectors $\hat{\boldsymbol{\sigma}}$, $\boldsymbol{q}_1$, and $\boldsymbol{q}_2$ be complanar. For two-dimensional problems (three-dimensional trajectories in Il'yushin's vector space), this means that the stress vector $\boldsymbol{\sigma}$ lies in the osculating plane of the strain path. In this case, the approximating relation can be written in the form

$$\boldsymbol{\sigma} = P_1\boldsymbol{q}_1 + P_2\boldsymbol{q}_2, \tag{1.1}$$

where $P_1 = |\boldsymbol{\sigma}|\cos\varphi$ and $P_2 = -|\boldsymbol{\sigma}|\sin\varphi$ (φ is the variable angle between the vectors $\boldsymbol{\sigma}$ and $d\ni/ds$). Generally speaking, this assumption is justified only for two-dimensional trajectories.

Thus, by defining the law of variation of the convergence angle φ and also the dependence $\sigma \sim s$, we can set the constitutive relation for the plastic material. The function $|\boldsymbol{\sigma}| = F(s)$, where $F(s)$ is the strain path, can be

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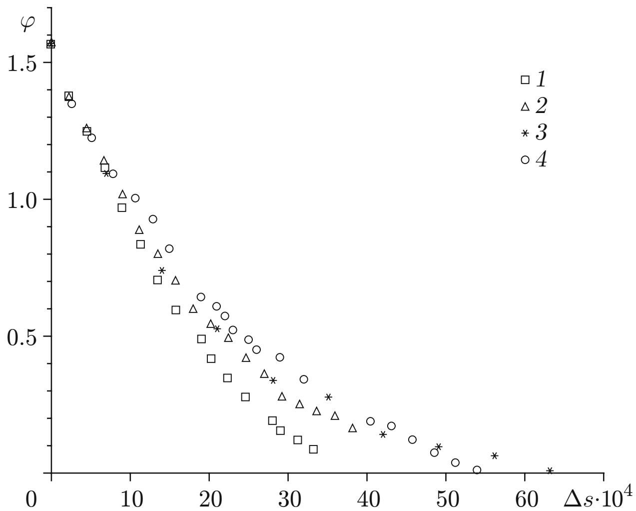

Fig. 1. Characteristic of the convergence angle φ between the vectors $\boldsymbol{\sigma}$ and $d\ni/ds$ under loading along a trajectory in the form of two-element trajectories ($\theta = 90°$) with different values of prior strains (before the bend point): $\varepsilon = 42.68 \cdot 10^{-4}$ (1), $70.5 \cdot 10^{-4}$ (2), $53.88 \cdot 10^{-4}$ (3), and $76.0 \cdot 10^{-4}$ (4).

determined from experiments on pure shear or simple tension. The convergence angle φ substantially depends on the strain-path curvature. For simple loading processes, we have $\varphi = 0$. Processes of medium and small curvature, however, are the most reproducible ones under static loading.

As it is assumed that strain-path twisting can be neglected, φ in the general case is a functional only of the strain-path curvature χ, trace of delay of the vector properties of the material λ, and length of the strain-path arc s.

Experiments with isotropic metals (St.3 steel, LS-59 brass, St.45 steel) following trajectories with a bend point confirmed the universal character of the quantity φ/θ (θ is the bend angle of the strain path). The same conclusions were made on the basis of the experimental data obtained in [3]. Figure 1 shows the dependence $\varphi(\Delta s)$ for St.3 steel, which was obtained in experiments with a bend angle $\theta = 90°$ and different values of prior strains (before the bend point). On the average, the scatter of the experimental data is seen to be within the measurement error (8%). A large scatter is observed in the case of moderate values of prior strains before the bend point. The reason is the use of difference relations with a small number of experimental nodes in plotting this dependence.

Based on the experiments performed with steel and brass of different types, the following approximation of the functional φ was proposed [4]:

$$\varphi = \theta / \exp\,(2.7725 \Delta s / \lambda) \tag{1.2}$$

(Δs is the length of the arc of the strain path after the bend point). The main mechanism of plastic deformation in these materials is dislocation gliding inside the grains. For other mechanisms of the flow, verification of the correspondence between this approximation and real vector properties of the material requires additional experimental research.

The approximation of the convergence angle in the form (1.2) agrees with the principle of delay, i.e., it obeys the following conditions:

$$\varphi = \theta \quad \text{for} \quad \Delta s = 0, \qquad \varphi \approx 0 \quad \text{for} \quad \Delta s = \lambda, \qquad \varphi \to 0 \quad \text{for} \quad \Delta s \to \infty.$$

The exponential form of the curve $\varphi \sim s$ was also predicted previously by other researchers.

As the stress vector is assumed to lie in an osculating plane, we take this functional for the convergence angle φ in the form [5]

$$\varphi = \int_0^s K(s, x) \chi(x)\, dx. \tag{1.3}$$

The strain path curvature is assumed to be known, and the functional kernel is determined from the results of experiments with complex loading. The kernel $K(s,x)$, which is considered as a universal function of the material, can be determined from the results of experiments with the use of straight-line paths with a bend point. The curvature of such paths equals zero. An exception is the bend point where the curvature tends to infinity. Thus, we obtain

$$\varphi = \int\limits_0^s K(s,x)\delta(x,\xi)\,dx = K(s,\xi),$$

where $\delta(x,\xi)$ is the Dirac delta function.

With allowance for the information given above, the kernel for moderate curvatures, which do not imply the presence of bend points, should be chosen in the form

$$K(s,x) = \exp\,(2.7725(x-s)/\lambda). \tag{1.4}$$

Thus, relations (1.1), (1.3), and (1.4) are the constitutive equations of plasticity for an isotropic material, which were derived on the basis of experimental data. The plasticity functional implies the presence of a kernel, which depends both on the material properties and on the strain path length.

As a result of forging, cogging, or other mechanical forcing, polycrystalline metals acquire essential anisotropic features, which cannot be neglected. Under certain treatment, crystallites lose their initial equiaxial form, and the metal acquires a fibrous structure and cold hardening. The experimental data obtained are used below to describe texture-possessing materials.

2. Model of Plasticity for a Transversely Isotropic Medium. It should be noted that the model considered in the present paper is an ideal continuous medium with the molecular or crystalline structure of the material being ignored.

All tensors of the second rank, in particular, stress and strain tensors, can be divided into the spherical and deviatoric parts. The spherical tensor corresponds to all-sided compression, and the deviator corresponds to shear stresses and displacements. In an isotropic medium, the hydrostatic pressure does not affect the shape of the body, in contrast to an anisotropic medium.

For an anisotropic medium, it seems reasonable to divide tensors of the second rank into mutually orthogonal parts, i.e., present them in a certain orthogonal reference frame. Each part should retain its shape with respect to a group of transformations characterizing the class of anisotropy.

For a transversely isotropic medium, we can present the tensor of the second rank in an orthogonal Cartesian coordinate system (direction 3 is the axis of transverse isotropy) [6]:

$$t_{ij} = 0.5(t_{11}+t_{22})(\delta(i,1)\delta(j,1)+\delta(i,2)\delta(j,2)) + t_{33}\delta(i,3)\delta(j,3) + T_{ij} + V_{ij}.$$

Here

$$T_{ij} = t_{ij} + 0.5(t_{11}+t_{22})(\delta(j,3)\delta(i,3)-\delta(i,j)) + t_{33}\delta(j,3)\delta(i,3) - (t_{i3}\delta(j,3)+t_{3j}\delta(i,3)),$$

$$V_{ij} = t_{i3}\delta(j,3) + t_{3j}\delta(i,3) - 2t_{33}\delta(j,3)\delta(i,3).$$

All parts of the tensor in this expansion are pairwise orthogonal, and each of them retains its shape after the transformation with respect to the transverse isotropy axis. The quantities T_{ij} and V_{ij} corresponding to the stress and strain tensors are denoted, respectively, by P_{ij}, Q_{ij} and p_{ij}, q_{ij}.

In the theory of elasticity, the invariants are related as

$$\tilde\theta = \frac{1-\nu}{E}\,\tilde\sigma - \frac{2\nu'}{E'}\,\sigma_{33}, \qquad \varepsilon_{33} = -\frac{\nu'}{E'}\,\tilde\sigma + \frac{1}{E'}\,\sigma_{33}, \qquad p = \frac{1}{2G}\,P, \qquad q = \frac{1}{2G'}\,Q,$$

where $\tilde\theta = \varepsilon_{22}+\varepsilon_{11}$, $\tilde\sigma = \sigma_{22}+\sigma_{11}$, $P = \sqrt{P_{ij}P_{ij}}$, $p = \sqrt{p_{ij}p_{ij}}$, $Q = \sqrt{Q_{ij}Q_{ij}}$, $q = \sqrt{q_{ij}q_{ij}}$, G and G' are the shear moduli in the isotropy plane and in an arbitrary plane perpendicular to the isotropy plane, ν is Poisson's ratio characterizing transverse compression in the isotropy plane under tension, ν' is Poisson's ratio for the case of tension in the direction normal to the isotropy plane, and E and E' are Young's moduli under tension-compression in the directions of the isotropy plane and the plane normal to the latter, respectively.

Under plastic strains, the dependences between the quantities ε_{33}, σ_{33}, $\tilde{\sigma}$, and $\tilde{\theta}$ with allowance for plastic incompressibility of the material remain unchanged, and the dependences between the quantities P_{ij}, Q_{ij}, p_{ij}, and q_{ij} have to be determined.

As each of the quantities P_{ij}, Q_{ij}, p_{ij}, and q_{ij} has two independent components, we can introduce a two-dimensional Il'yushin's space $\ni^2$ for each of them, by analogy with the isotropic case. The difference between these two cases is as follows. For an isotropic material, a five-dimensional vector space $\ni^5$ was used in the general case both for stresses and for strains. For a transversely isotropic medium, we introduce four two-dimensional vector spaces: two for stresses (P^2 and Q^2) and two for strains (p^2 and q^2). For the processes of loading in two-dimensional vector spaces, we can use the isotropy postulate, because all transformations in these spaces are possible only in the isotropy plane.

With allowance for the reasoning given above, we obtain the following relations between the stress and strain vectors:

$$\boldsymbol{P} = L_{11}\boldsymbol{q}_{11} + L_{12}\boldsymbol{q}_{12}, \qquad \boldsymbol{Q} = L_{21}\boldsymbol{q}_{21} + L_{22}\boldsymbol{q}_{22}. \tag{2.1}$$

Here $L_{11} = |\boldsymbol{P}|\cos\varphi_1$, $L_{12} = -|\boldsymbol{P}|\sin\varphi_1$, $L_{21} = |\boldsymbol{Q}|\cos\varphi_2$, $L_{22} = -|\boldsymbol{Q}|\sin\varphi_2$, and $\boldsymbol{q}_{ij}$ are the Frenet vectors in the spaces p^2 and q^2. The components of the stress and strain vectors are determined as

$$P_1 = \sqrt{2}\,P_{11}, \quad P_2 = \sqrt{2}\,P_{12}, \quad p_1 = \sqrt{2}\,p_{11}, \quad p_2 = \sqrt{2}\,p_{12},$$

$$Q_1 = \sqrt{2}\,\sigma_{13}, \quad Q_2 = \sqrt{2}\,\sigma_{23}, \quad q_1 = \sqrt{2}\,\varepsilon_{13}, \quad q_2 = \sqrt{2}\,\varepsilon_{23}$$

(φ_1 is the angles of convergence of the vectors $\boldsymbol{P}$ and $\boldsymbol{p}$ and φ_2 are the angles of convergence of the vectors $\boldsymbol{q}$ and $\boldsymbol{Q}$).

The angles of convergence in the spaces p^2 and q^2 are introduced as

$$\varphi_1 = \int_0^p K_1(p,x)\chi_1(x)\,dx, \qquad \varphi_2 = \int_0^q K_2(q,x)\chi_2(x)\,dx. \tag{2.2}$$

The kernels are taken in the form

$$K(p,\xi) = \exp\left(-2.7725(p-\xi)/\lambda_1\right), \qquad K(q,\zeta) = \exp\left(-2.7725(q-\zeta)/\lambda_2\right). \tag{2.3}$$

In studying the processes of complex loading, one has to know additional properties of materials, such as the trace of delay in the isotropy plane λ_1 and in the plane λ_2 normal to the latter. Thus, the total number of characteristics of a transversely isotropic body is seven.

Thus, Eqs. (2.1)–(2.3) define the model of plasticity under complex loading of a transversely isotropic material. After some transformations, these equations can be written in a differential form as

$$\frac{d\boldsymbol{P}}{d\boldsymbol{p}} = \frac{\boldsymbol{P}}{|\boldsymbol{P}|}\Big(\frac{dP}{dp} - P\cos\varphi_1\,\frac{\chi_{11}-\dot{\varphi}_1}{\sin\varphi_1}\Big) + \boldsymbol{q}_{11}P\,\frac{\chi_{11}-\dot{\varphi}_1}{\sin\varphi_1} - P\chi_{12}\sin\varphi_1\boldsymbol{q}_{13},$$

$$\frac{d\boldsymbol{Q}}{d\boldsymbol{q}} = \frac{\boldsymbol{Q}}{|\boldsymbol{Q}|}\Big(\frac{dQ}{dq} - Q\cos\varphi_2\,\frac{\chi_{21}-\dot{\varphi}_2}{\sin\varphi_2}\Big) + \boldsymbol{q}_{21}Q\,\frac{\chi_{21}-\dot{\varphi}_2}{\sin\varphi_2} - Q\chi_{22}\sin\varphi_2\boldsymbol{q}_{13}.$$

Here Q, q, P, and p are the lengths of the corresponding loading and strain paths in the spaces of transverse isotropy.

If we assume that $\chi_{12} = \chi_{22} = 0$, we obtain equations based on the complanarity hypothesis. Then, if the angles φ_1 and φ_2 are small, we obtain equations based on the classical Il'yushin's theory:

$$\frac{d\boldsymbol{P}}{d\boldsymbol{p}} = \frac{\boldsymbol{P}\boldsymbol{q}_{11}}{|\boldsymbol{P}|}\,\frac{\boldsymbol{P}}{|\boldsymbol{P}|}\,(M_1 - N_1) + \boldsymbol{q}_{11}N_1, \qquad \frac{d\boldsymbol{Q}}{d\boldsymbol{q}} = \frac{\boldsymbol{Q}\boldsymbol{q}_{21}}{|\boldsymbol{Q}|}\,\frac{\boldsymbol{Q}}{|\boldsymbol{Q}|}\,(M_2 - N_2) + \boldsymbol{q}_{21}N_2. \tag{2.4}$$

By varying the coefficients M_1, N_1, M_2, and N_2, we obtain different variants of the plasticity theory for transversely isotropic media.

For a certain choice of the coefficients in Eq. (2.4), we obtain the theory constructed in [7]. Thus, the model proposed in the present paper coincides in a particular case with the classical model of plasticity.

This model was constructed for a standard material with a low compressibility on the basis of experiments with two-dimensional trajectories; therefore, this model is proposed to be used for solving two-dimensional problems with allowance for complex active loading.

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